Grading guide, Pricing Financial Assets, August 2017

1. Let the time t spot exchange rate for a foreign currency be S_t, denoting the value of one unit of the foreign currency as measured in the domestic currency.

   Assume that the exchange rate can be be modelled (under the original probability measure $\mathbb{P}$) by the geometric Brownian motion
   $$dS = \mu_S S dt + \sigma_S S dz$$
   where μ_S and $\sigma_S > 0$ are constants, and where dt and dz are the standard shorthand notations for a small time-step and a Brownian increment.

   Assume that the (continuously compounded) domestic and foreign risk free interest rates are constants r and r_f, respectively.

   (a) What will the drift rate of the exchange rate be under the domestic risk neutral probability measure ($\mathbb{Q}$) and a no-arbitrage assumption? Comment on the influence of the difference of domestic and foreign interest rates on the result.

   (b) Consider at time $t < T$ a forward contract on the foreign currency for payment and delivery at time T. What is the arbitrage free forward price F_t^T?

   (c) Use Ito's lemma to show that the volatility of F_t^T goes to zero as time t approaches maturity T.

Solution:

(a) The drift rate will be $(r - r_f)S$, i.e. determined by the domestic risk free interest rate less the foreign interest rate (analogous to a stock paying a continuous dividend rate). Under $\mathbb{Q}$ the currency with a high interest rate is expected to depreciate at a rate corresponding to the interest rate differential (the uncovered interest rate parity).

(b) Consider two strategies to end with the foreign currency at time T: A) Enter into a forward contract at time t. You pay the agreed forward price F_t^T at time T. B) Borrow $S_t \mathrm{e}^{-r_f(T-t)}$ at the domestic risk free rate and buy the currency spot placing on deposit at the foreign rate. At T you have accumulated one unit of the foreign currency at the deposit. You repay the loan that have accumulated to $S_t \mathrm{e}^{(r-r_f)(T-t)}$, In the absence of arbitrage opportunities the forward price contracted at t thus must be $F_t^T = S_t \mathrm{e}^{(r-r_f)(T-t)}$, (the covered interest rate parity).

(c) For ease of notation let $\alpha = (r - r_f)(T - t)$. Since $\frac{\partial F_t^T}{\partial S_t} = \alpha S_t^{\alpha-1}$ we have by Ito's lemma that the volatility of F_t^T is $\frac{\partial F_t^T}{\partial S_t}\sigma_S S_t = \alpha \sigma_S S_t^{\alpha}$. Since α goes to zero as t goes to T, we have the required result.

2. Let S_t be the price in the domestic currency of one unit of a foreign currency. Assume that there are constant risk free, continuously compounded interest rates of r and r_f in the domestic and foreign currencies, respectively.

   Consider a derivative with price $V(S, t)$ as some function of the current exchange rate S_t and time t (and further implicit parameters).

   (a) Define and interpret the Delta, Gamma and Theta of the derivative.

   (b) Let $c(S, K, T, r, r_f)$ and $p(S, K, T, r, r_f)$ be the price at time $t = 0$ of a European call and a European put, respectively, on the currency with the same strike K and expiry T. Derive the call-put-parity.

(c) Use the call-put-parity to find a relationship between the Deltas of the call and put. Repeat this for Gamma.

(d) Suppose a portfolio of the foreign currency and/or derivatives on the currency is Delta-neutral, and that there are no arbitrage possibilities. Let the value of the portfolio be $\Pi(S,t)$. What can we say about the relation between the Theta and Gamma of the portfolio?

Solution:

(a) The Delta, Gamma and Theta are defined as for derivatives on stocks, see Hull, chapter 18 (8th ed.).

(b) A portfolio of the long call and a short put can be seen as a forward contract on the currency with K as the forward price. This contract has current value of $S_0 e^{-r_f T} - KS_0 e^{-rT}$ (you may also use an arbitrage argument to find this). Thus barring arbitrage $c + Ke^{-rT} = p + S_0 e^{-r_f T}$.

(c) This follows directly, e.g. for the case of the Delta $\Delta_c = \Delta_p + e^{-r_f T}$. Then you also see that the Gammas are identical. There is a table 18.6 i Hull (8th ed), where these relationships can be found if you just interpret the dividend yield as the foreign interest rate.

(d) It follows from the fundamental Black-Scholes-Merton PDE (Hull formula 16.6 (8th ed)) that

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - rV = 0$$

Substituting the definitions in and using that the delta is zero, you get

$$\Theta + \tfrac{1}{2}\sigma^2 \Gamma = rV$$

Thus beyond the risk free rate the Theta and the Gamma works in opposite directions. For a portfolio that is long Gamma (e.g. a long option) you will in this sense be short Theta - in this case you earn from your Gamma when there is price movement, but the loss from the Theta dominates when there is not (Hull, section 18.6 (8th ed.)).

3. The LMM-model describes the evolution of the term structure of simple interest rates. Let $F_k(t)$ be the (simple) forward rate between times t_k and t_{k+1} that can be contracted for at time $t \le t_k < t_{k+1}$, and let τ_k be the compounding period between t_k and t_{k+1}. Let $P(t, t_k)$ denote the price at t of a zero coupon bond that matures at time t_k.

(a) What is the relationship between a particular forward rate F_k and prices of zero coupon bonds?

(b) Assume that the forward rates are driven by one factor. What choice of numeraire will lead to the forward rate $F_k(t)$ being a martingale, e.g. of the form

$$dF_k(t) = \zeta_k(t) F_k(t) dz$$

where dz is the standard short hand notation for the change of a Brownian motion.

(c) In the development of the LMM-model a rolling numeraire is used. Explain in general terms how this is constructed (explicit formulas are not required, but you should discuss change of numeraires).

Solution:

(a) We define or see as an arbitrage relationship

$$1 + \tau_k F_k = \frac{P(t, t_k)}{P(t, t_{k+1})}$$

(b) The numeraire must be the zero coupon bond that is used for discounting when there is the payment on the forward rate contract, i.e. the numeraire $P(t, t_{k+1})$, cf. Hull (31.7, 8th ed). This is the choice that is said to be forward risk neutral with respect to this numeraire.

(c) In the LMM-model there is a rolling change of numeraire where we use the next reset date of discretely compounded (Libor-) rates. If current date is t we use the notation $m(t)$ for the next reset date. Using $P(t, t_{m(t)})$ as a numeraire the forward rate will not be a martingale if $m(t) < k$. But we know that the change in drift is determined by the difference in volatilities of the two numeraires (Hull chapter 27 and 31). This constitutes a satisfactory answer. One may however also note that if we assume that zero coupon prices are driven by geometric Brownians motions, and let v_t denote the volatility of the zero coupon bond maturing at t we have

$$dF_k(t) = \zeta_k(t)[v_{m(t)}(t) - v_{k+1}(t)]F_k(t)dt + \zeta_k(t)F_k(t)dz$$